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LETTER TO THE EDITOR

# On perturbed and supersymmetric quantum KdV equations

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**Abstract.** We have deduced the quantum version of the perturbed and supersymmetric KdV equation by extending an approach by Kupershmidt. Our approach also yields some quantum conservation laws in both cases. In the case of the perturbed KdV equation we show that its quantum version arises by demanding higher-order conformal invariance.

In a recent communication it was demonstrated by Kupershmidt *et al* [1] that a quantum version of the KdV equation can be deduced via the formalism of string theory and operator product expansion (OPE). The derivation is very important [2] in exhibiting the important role of the central extension term in the construction of infinite dimensional integrable systems. Prior to [1] there was attempt to construct an 'operator' version of the KdV problem by Bogoliuobsky [3], but this could not be called a quantum mechanical one. Here in this letter we show that the perturbed KdV problem discussed by Kodama [4] and the super-KdV equation studied by Kupershmidt [5], Manin [6] and Mathieu [7] possess quantum extensions, the integrability of which is guaranteed by the existence of an infinite number of conserved quantities.

Let us start with the perturbed KdV equation written as:

$$u_t = h_0 + \varepsilon h_1 \tag{1}$$

with $h_0 = u_{xxx} + 6uu_x$

$$h_1 = a_1 u_{5x} + a_2 uu_{3x} + a_3 u_{1x} u_{2x} + a_4 uu_x^2. \tag{2}$$

It was shown in [4] that the nonlinear system (2) can be transformed into a Hamiltonian system by a Birkhoff-like transformation up to terms of order

$$u_t = \{u, H\} + \mathrm{O}(\varepsilon^2) \tag{2}$$

$$\begin{aligned} H[u] &= H_0[u] + \varepsilon H_1[u] \\ &= \int_{-\infty}^{\infty} (u^3 - \tfrac{1}{2}u_x^2)\,\mathrm{d}x + \varepsilon \int_{-\infty}^{\infty} (b_3 uu_x^2 + b_4 u^4)\,\mathrm{d}x \end{aligned} \tag{4}$$

where the Poisson bracket in (2) is defined by:

$$\begin{aligned} \{F, G\}[u] &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F\left[u + \varepsilon g \frac{\mathrm{D}G}{\mathrm{D}u}\right]_{\varepsilon=0} \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{D}F}{\mathrm{D}u} g \frac{\mathrm{D}G}{\mathrm{D}u}\,\mathrm{d}x \end{aligned} \tag{5}$$

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where

$$g=\partial_x+\varepsilon[b_1\partial_x^3+b_2(\partial_x u+u\partial_x)] \tag{6}$$

$(a_1, a_2, a_3, a_4)$ being related to $(b, i=1,2,3,4)$ by some algebraic relations.

Now the canonical form of the quantum KdV problem as given in [1] is;

$$\dot{T}=[T,H] \qquad H=\oint (TT)\,\mathrm{d}z \tag{7}$$

where the normal ordering is defined via

$$(AB)(z)=\oint \frac{\mathrm{d}x}{x-z}A(x)B(z) \tag{8}$$

and the KdV is easily seen to arise from (6) through the OPE

$$T(z)T(w)=\frac{c/2}{(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{T'(w)}{(z-w)}+\text{regular terms.} \tag{9}$$

We now recapitulate the basics of the procedure for deducing the OPE of the energy momentum tensor through conformal invariance. Suppose we use complex coordinates $z=x_1+\mathrm{i}x_2$; $\bar{z}=x_1-\mathrm{i}x_2$ and the infinitesimal transformation for any function $A(z,\bar{z})$ is written as

$$A(z,\bar{z})=\oint \mathrm{d}z'\;\varepsilon(z')[T(z'),A(z',\bar{z})] \qquad |z'|=|z|. \tag{10}$$

From the definition any primary field transforms as

$$A(z,\bar{z})\to\left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{\Delta}\left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}z}\right)^{\bar{\Delta}}A(w,\bar{w}) \tag{11}$$

$$\Delta+\bar{\Delta}=\text{conformal field dimension}$$

$$\Delta=\bar{\Delta}=\text{conformal spin dimension.}$$

If $w(z)$ is near $z$, that is

$$w(z)=z+\varepsilon(z)$$

then

$$A(z',z)=A'(z,\bar{z})-A(z,\bar{z})$$

and

$$A'(z,z)=(1+\varepsilon')^{\Delta}(1+\bar{\varepsilon}')^{\bar{\Delta}}V(w,\bar{w})$$

$$\varepsilon'(z)=\frac{\mathrm{d}\varepsilon(z)}{\mathrm{d}z}.$$

Instead of the usual procedure of keeping only first-order terms in $\varepsilon$, we also retain terms of the order of $\varepsilon^2$, so we get

$$A(z,\bar{z})=\{\varepsilon A'(z,\bar{z})+\Delta\varepsilon' A(z,\bar{z})\}+\left\{\frac{\varepsilon^2}{2}A''(z,\bar{z})+\Delta\varepsilon\varepsilon' A'(z,\bar{z})+\frac{\Delta(\Delta-1)}{2}\varepsilon'^2 A(z,\bar{z})\right\}.$$

By Cauchy's theorem we get

$$A(z,\bar z)=\oint \mathrm{d}z'\,\varepsilon(z')\left\{\frac{\Delta}{(z'-z)^2}+\frac{1}{z'-z}\partial z\right\}A(z,\bar z)$$
$$+\oint \mathrm{d}z'\,\varepsilon^2(z')\left\{\frac{\Delta(\Delta-1)}{2}+\frac{1}{(z'-z)^4}+\frac{\Delta\partial z}{(z'-z)^3}\right\}+\frac{\partial z^2}{(z'-z)^2}A(z,\bar z). \tag{12}$$

Applying (9) to the case of $T(z,\bar z)$ with $\Delta=2$, we can write

$$T(z')T(z)=\frac{c/2}{(z'-z)^4}+\frac{2T(z)}{(z'-z)^2}+\frac{T'(z)}{z'-z}$$
$$+\varepsilon\left\{\frac{T(z)}{(z'-z)^4}+\frac{2T'(z)}{(z'-z)^3}+\frac{T''(z)}{(z'-z)^2}\right\}+\dots. \tag{13}$$

We now proceed to derive the perturbed quantum KdV equation with the help of equation (13). Let us consider the Hamiltonian

$$H=\oint (TT)(z)\,\mathrm{d}z+\varepsilon\oint (TTT)(z)\,\mathrm{d}z \tag{14}$$

and evaluate

$$\dot T=[T,H] \tag{15}$$

via (13). To evaluate (15) we use the rule:

$$T(z)(TT(w))=\oint\frac{\mathrm{d}x}{x-w}\{\underbrace{T(z)T}(x)T(w)+T(x)\underbrace{T(z)T}(w)\}$$

whence we obtain (after identifying $\varepsilon$ in (13) with $\varepsilon$ in (12)) up to first order in $\varepsilon$:

$$\dot T=\frac{1}{6}(1-c)T'''(z)-3(TT)'(z)$$
$$+\varepsilon\left\{\frac{9}{40}\left(1-\frac{c}{3}\right)T'''''(z)+\left(1-\frac{c}{4}\right)(TT)'''-5(TTT)'\right\} \tag{16}$$

which contains terms of all types as in Kodama's equation (2), a $\varepsilon$ free term giving the quantum KdV equation of Kupershmidt.

We now consider the supersymmetric KdV equation. To proceed we consider the superspace coordinate $z$ and super-energy momentum tensor

$$S=G+\theta T \tag{17}$$

where $\theta=$ Grassmannian variable. If $D$ denotes the super-covariant derivative, $D=\theta\,\partial/\partial z+\partial/\partial\theta$; then

$$H=\int_0^{2\pi}\mathrm{d}\hat z(SDS). \tag{18}$$

In the usual coordinates we write

$$\begin{aligned}\dot T&=[T,H]\\ \dot G&=[G,H]\end{aligned} \tag{19}$$

$$H=\oint \mathrm{d}z\{(TT)(z)-(GG')(z)\} \tag{20}$$

where we have used $\int \theta \, d\theta = 1, \int d\theta = 0$. Writing explicitly (19) can be put into the form:

$$\begin{aligned} \dot{T} &= -\oint T(z)(TT)(w)\,dw + \oint T(z)(GG')(w)\,dw \\ \dot{G} &= -\oint G(z)(TT)(w)\,dw + \oint G(z)(GG')(w)\,dw. \end{aligned} \tag{21}$$

We now use the operator product expansion given by the superconformal algebra

$$\begin{aligned} T(z)T(w) &\sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{T(w)}{(z-w)} + \dots \\ T(z)G(w) &\sim G(z)T(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{G'(w)}{z-w} + \dots \\ G(z)G(w) &\sim \frac{2c}{(z-w)^3} + \frac{2T(w)}{z-w} + \dots . \end{aligned} \tag{22}$$

Using the normal ordering via contour integrals we get

$$\begin{aligned} T &= (1/2 - c/6)T''' - 3(TT)' + 3(GG')' \\ G &= -9/4G'''(z) + 2(GT)'. \end{aligned} \tag{23}$$

It is interesting to compare the coefficients in (23) with that of the classical counterpart deduced by Mathieu.

Since these equations are completely integral, they do possess an infinite number of conserved quantities. It is possible to write a few of them as follows:

$$\begin{aligned} H_2 &= \oint dz[(TT)(z) - (GG')(z)] \\ H_3 &= \oint dz\,[T(TT)(z) - 2G(G'T)(z)] + \lambda[(T'T')(z) - (G'G''')(z)]\} \end{aligned} \tag{24}$$

in the super case and

$$\begin{aligned} C_2 &= \oint (TT)\,dz + \varepsilon \oint (TTT)(z)\,dz \\ C_3 &= \oint [T(TT)(z) - \tfrac{1}{12}(2+C)(T'T')(z)]\,dz + \varepsilon \oint [T(T'T') + (T(T(TT))(z)]\,dz \end{aligned} \tag{25}$$

in the perturbed KdV case.

It is to be noted that these integrals of motion must be in involution, that is $\{C_i, C_j\} = 0$, which can be verified by the procedure laid down by Bais *et al* [9].

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